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MRC Technical Summary Report #2117

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# THE BEHAVIOR OF SPHERICALLY SYMMETRIC EQUILIBRIA NEAR INFINITY

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## ABSTRACT

The study of the radially symmetric equilibria of a non-linear diffusion equation in several space dimensions leads to an ordinary differential equation. Under the hypothesis that the reaction terms are in gradient form, conditions are found which imply that solutions are asymptotically constant at infinity. As an application it is shown that the all spherically symmetric solutions of the sine-Gordon equation are asymptotically constant (and consequently bounded).

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## SIGNIFICANCE AND EXPLANATION

Diffusion - reaction equations approximately govern the behavior of chemical reactions, population changes etc. The analysis of such equations begins with a description of those solutions that are constant in time. Frequently, something about the number and type of such solutions can be found without actually finding the solution itself. In such a case further information about the solution might follow from general considerations.

This work provides a *general means of determining* that certain equilibrium solutions must be approximately constant for large values of the space variables.

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The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

# THE BEHAVIOR OF SPHERICALLY SYMMETRIC EQUILIBRIA NEAR INFINITY

C. Conley

§1. The problem considered here arose in the study of spherically symmetric equilibria of a non-linear diffusion reaction equation in the special form:

$$\partial u/\partial t = D\,\Delta\, u + \nabla\, F(u)\ , \tag{1}$$

where $u$ is an m-vector valued function of the n-vector $x$ and scalar $t$, $D$ is a positive definite symmetric $m \times m$ matrix and $\Delta$ is the Laplacian operator in $\mathbf{R}^n$ acting component-wise on $u$. The specialization is in the assumption that the non-linear terms have the form $\nabla F(u)$ where $F$ is a (smooth) scalar-valued function on $\mathbf{R}^n$ and $\nabla$ is the gradient operator.

The equation for radially symmetric equilibria of (1) can be written as a first order system of ordinary differential equations:

$$\begin{aligned} du/dr &= D^{-1}v \\ dv/dr &= -(n-1)v/r - \nabla F(u). \end{aligned} \tag{2}$$

In these equations $u$ and $v$ are m-vector valued functions of $r$. The aim is to give conditions under which a solution tends to a constant as $r$ tends to infinity. Since the constant, say $(u_\infty, v_\infty)$, must be a rest point of the "limit" equation (i.e. (2) with the $1/r$ term absent) $v_\infty$ must be zero and $u_\infty$ must be a critical point of $F$.

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The result is that if the limit set of the solution is compact and either contains a minimum of $F$ or contains only maxima of $F$, then it must contain only rest points. Since it is also connected, it must then be contained in some component of the rest point set. The case where the limit set contains a minimum is trivial but the case with maxima (Theorem 1 of §2) requires a little more argument.

It is not shown that compactness of the limit set implies it contains only rest points, even though this seems the obvious thing to expect. The main point is that the function $H(u,v) = (v,D^{-1}v)/2 + F(u)$ is strictly decreasing on non-constant solutions of (2). If that equation were autonomous, it would follow that all bounded solutions tend to rest points. However, in the present case the rate of decrease is $(v,D^{-1}v)/r$ and without further work, it can only be concluded that a bounded solution tends to a set of solutions of the limit (Hamiltonian) system and that this set must be contained in some one level surface of $H$.

However, since $H$ decreases on solutions, some conditions under which the limit set must be compact (i.e. conditions enabling application of the theorem) are obvious. For example, if the initial conditions lie in a (compact) component of a set of the form $\{(u,v) | H(u,v) \leq h\}$ (where $h$ is any constant) then the full solution curve must lie in the same component.

Also, if $F(u)$ is periodic in $u$ with a compact periodic domain then the compactness hypothesis is satisfied on the identification space and the theorem applies. In particular, it follows that radially symmetric equilibria of the sine-Gordon equation $u_{tt} = \Delta u + \sin u$ are bounded and asymptotically constant as $r$ tends to $\infty$. (cf. §3).

Other solutions which are bounded as $r$ tends to infinity come out of the study of traveling wave solutions of (1). This comes from a correspondence between these waves and the radially symmetric equilibria. This will be described in another paper where the present theorem on asymptotic behavior is used to refine the existence results proved there.

It seems to be true that if a traveling wave is to be stable, it's limits at infinity should be maxima of $F$. If this is the case, Theorem 1 applies in the case of stable waves.

Theorem 1 is proved in §2 and Theorem 2, concerning periodic F, in §3.

## §2. Theorem 1.

By introducing $\rho = 1/r$ the equations (2) can be rewritten in the autonomous form:

$$
\begin{aligned}
u' &= D^{-1}v \\
v' &= -(n-1)\,\rho v - \nabla F(u) \\
\rho' &= -\rho^2
\end{aligned}
\tag{3}
$$

where $' = d/dr$. It is assumed that $n$ is greater than 1.

The rest points of (3) are all of the form $(u_0, 0, 0)$ where $\nabla F(u_0) = 0$ ; then, they are in one-one correspondence with the critical points of F.

Suppose $(u(r), v(r), \rho(r))$ is a solution of (3) with limit set $\Omega$ as $r \to \infty$. Note that $\rho$ is identically zero on $\Omega$. The aim is to prove $v$ is zero on $\Omega$, at least under some conditions. Let $\Omega_u$ be the set of u-coordinates of points in $\Omega$.

Since the derivative of $H(u,v)$ along solutions of (3) is $-(n-1)\rho(v,D^{-1}v)$ (with $n-1 > 0$) H is non-increasing on solutions and must be constant on $\Omega$. Thus $\Omega$ lies in a level set of H, say $H=h$. Since $H = (v,D^{-1}v) + F(u)$, this implies $\Omega_u$ is contained in the set $F \leq h$. When F is replaced by $F-h$, the equation doesn't change, so it can be assumed that $h = 0$. Then it follows that F is non-positive on $\Omega_u$.

In particular, it follows that if $\Omega_u$ contains a (strict) minimum of F then $\Omega$ consists of just one rest point. More generally, suppose C is a (connected) set in the zero level of F and is the intersection of neighborhoods U such that the restriction of F to the boundary of U is positive (so C consists of critical points of F). Then $\Omega_u$ is either disjoint from C or contained in C.

The theorem below pertains to maxima of F.

Theorem 1. Let $(u(r), v(r), \rho(r)$ be a solution of (3) with limit set $\Omega$ and let $\Omega_u$ be the set of u-coordinates of points in $\Omega$.

Then $\Omega$ is contained in a level surface of $H$ which can be assumed to be the zero-level. If $\Omega$ is compact, $\Omega_u$ is a connected compact set containing a critical point of $F$.

Let $C$ be the set of critical points in $\Omega_u$. If $C$ admits a neighborhood $U$ such that $F$ is non-positive in $U$ then $\Omega$ consists of rest points. In particular, if $C$ contains a strict maximum of $F$, $\Omega$ consists of one rest point. //

Proof: In view of the paragraphs preceding the statement of the theorem it can be assumed that the solution $(u(r), v(r), \rho(r))$ has compact limit set $\Omega$ contained in the zero-level of $H$ and that $F|\Omega_u \leq 0$. It is well known that compact limit sets are connected and it follows that $\Omega_u$ is connected.

To see that $\Omega$ must contain a rest point, let $W$ be any open neighborhood of the rest point set and let $K$ be a compact neighborhood of $\Omega$. (These neighborhoods are in the $(u,v,\rho)$ space; in particular, $K$ contains a tail of the given solution). Given any solution $(\hat{u}(r),\hat{v}(r),\hat{\rho}(r))$ with initial values in $K\backslash W$, consider $\int_0^1(\hat{v},D^{-1}\hat{v})dr$. Were this zero, $\hat{v}$ would have to be identically zero on $[0,1]$ so that $\hat{u}$ would have to be constant and $\nabla F(\hat{u})$ zero. But this would require the solution to be a rest point and therefore in $W$ which it is not. Since $K\backslash W$ is compact it follows that there is a positive $\delta$ such that for solutions starting in $K\backslash W$, $\int_0^1(\hat{v},D^{-1}\hat{v})dr > \delta$.

Now consider the function $\rho^{-1}H$. The derivative of this function on solutions is $H - (v,D^{-1}v)$. On the given solution, $H$ decreases to zero

as $r$ tends to infinity, and $\rho^{-1}H$ is positive. Choose $r_0$ so that if $r > r_0$, $H < \delta/2$. Define a (possibly finite or even empty) sequence $r_1, r_2, \ldots$ as follows: let $r_1$ be the first (if any) $r > r_0$ such that $(u(r), v(r), \rho(r))$ is in $K \backslash W$. Having defined $r_n$, if $(u(r_n + 1)\ v(r_n + 1), \rho(r_{n+1}))$ is in $K \backslash W$, let $r_{n+1} = r_n + 1$. Otherwise, let $r_{n+1}$ be the first $r$ after $r_{n+1}$ such that $(u(r), v(r), \rho(r))$ is in $K \backslash W$.

Now from the choice of $r_0$ and the definition of the sequence it follows that

$$\rho^{-1}H(r_n+1) - \rho^{-1}H(r_n) = \int_0^1 [H(u(r_n+s), v(r_n+s)) - (v(r_n+s), D^{-1}v(r_n+s))] \le -\delta/2.$$

Thus if the solution were eventually outside $W$, $\rho^{-1}H$ would go to $-\infty$. But it is known to be positive. Therefore the solution enters $W$ for arbitrarily large $r$ and $\Omega$ therefore contains points in $W$. Since $W$ was an arbitrary neighborhood of the rest point set, $\Omega$ contains a rest point.

Now let $C$ be the set of critical points in $\Omega_u$ and let $U$ be any neighborhood of $C$ restricted to which $F$ is non-positive. Let $V$ be any neighborhood of the intersection of $\Omega$ and the rest point set such that the u-coordinates of points in $V$ lie in $U$.

Since $V$ contains all the rest points in $\Omega$ there is a neighborhood $W$ of the rest point set and an $r_* > 0$ such that if $r > r_*$ and $(u(r), v(r), \rho(r))$ is in $W$, then it is also in $V$.

Choose $r_0$ as before (that is so that if $r > r_0$ then $H$ evaluated on the solution is less than $\delta/2$) and also so that $r_0 > r_*$. Now when the solution is outside $W$, $\rho^{-1}H$ decreases at rate at least $\delta/2$. Also when the solution is in $W$ it is in $V$. Now in $V$,

$d(\rho^{-1}H)/dr = H - (v, D^{-1}v) = F(u) - \frac{1}{2}(v, D^{-1}v) \le 0$. Therefore $\rho^{-1}H$ is also

$$F(u) - \tfrac{1}{2}(v, D^{-1}v)$$

decreasing when the solution in in $V$. Since $\rho^{-1}H$ stays positive, it must eventually stay out of $\Omega \backslash W$ and so must stay in $V$. Since $V$ can be chosen to be an arbitrarily small neighborhood of the rest point set in $\Omega$, it follows that $\Omega$ itself consists of rest points. This proves the theorem.

## §3 Periodic F.

Suppose now that $F(u) = F(u_1,\dots,u_m)$ is periodic in each variable. Also assume $F$ is smooth and therefore bounded.

If $(u(r), v(r), \rho(r))$ is any solution of (3), $H(u(r), v(r)) = (v(r), D^{-1}v(r)) + F(u(r))$ is a non-increasing function so is bounded as $r$ goes to $\infty$. In fact this is even true when $n = 1$.

The bound on $H$ then implies that $v(r)$ is also bounded. However, it doesn't follow that $u$ is also bounded. For example, equation (4) below has unbounded solutions if $n = 1$.

$$\begin{aligned} u' &= v \\ v' &= -(n-1)\rho v - \sin u \\ \rho^1 &= -\rho^2 \end{aligned} \tag{4}$$

On the other hand, if the equations are viewed in the space obtained by identifying points whose u-coordinates differ by a period vector then the boundedness of $v(r)$ implies that all solutions admit a compact limit set. For example, the unbounded solutions of (4) (with $n = 1$) have a limit set which is a "circle" around the cylinder $[0,2\pi] \times R$ (which is the identified space).

Now the compactness obtained on going to the identified space allows the argument of the theorem to be applied in the case where $n > 1$. Doing so in the case of equation (4), it is found that, since all critical points of $F$ $(= -\cos u)$ are either maxima or minima, the theorem implies that all solutions tend to a rest point. But now on going back to the unidentified space, it follows that if $n > 1$, all solutions of (4) are bounded.

The general statement is:

Theorem 2: If $F(u)$ is smooth and periodic in all its variables then each solution of 3) with $n > 1$ has a compact limit set $\Omega$ in the identification space and the conclusions of Theorem 1 hold.

If $\Omega$ consists of critical points then the solution is bounded in the original space.

Corollary 1: In the case of one degree of freedom (i.e. $u, v \in R^1$) if $F$ has only maxima and minima then any bounded solution tends to a critical point. If $F$ is periodic, then all solutions are bounded; thus radially symmetric solution of the sine-Gordon equation tend to a constant as $r$ tends to $\infty$.

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